1. The sum of a number and 3 times another number is 12. Their product is a maximum. Find the numbers and the max product.

\[ \text{given: } \]
\[ x + 3y = 12 \]
\[ x = 12 - 3y \]
\[ x = 12 - 3(2) \]
\[ x = 6 \]

\[ \text{max/min: } \]
\[ P_{\text{max}} = xy \]
\[ P = (12 - 3y)y \]
\[ P_{\text{max}} = 12y - 3y^2 \]
\[ P' = 12 - 6y \]
\[ 0 = 6(2 - y) \]
\[ y = 2 \]

2. A rectangular piece of land is bordered on one side by a river. The other 3 sides are to be enclosed by 300 feet of fencing. What is the maximum area that can be enclosed?

\[ 2x + y = 300 \]
\[ y = 300 - 2x \]
\[ y = 300 - 2(75) \]
\[ y = 150 \text{ ft.} \]

\[ A_{\text{max}} = xy \]
\[ A_{\text{max}} = x(300 - 2x) \]
\[ A_{\text{max}} = 300x - 2x^2 \]
\[ A' = 300 - 4x \]
\[ 0 = 4(75 - x) \]
\[ x = 75 \text{ ft} \]

\[ A_{\text{max}} = (75)(150) \]
\[ A_{\text{max}} = 11250 \text{ ft}^2 \]

3. An open rectangular box is made by cutting 4 congruent squares from the corners of a square piece of cardboard and folding the sides up. The cardboard is 12 feet on each side. What are the dimensions of the box with the maximum volume?

\[ l = 12 - 2x \]
\[ w = 12 - 2x \]
\[ h = x \]

\[ l = 12 - 2(2) = 8 \text{ ft.} \]
\[ w = 12 - 2(2) = 8 \text{ ft.} \]
\[ h = 2 \text{ ft} \]

\[ V_{\text{max}} = lwh \]
\[ V_{\text{max}} = (12 - 2x)(12 - 2x)(x) \]
\[ V_{\text{max}} = 144 - 48x + 4x^2 \]
\[ V' = 144 - 96x + 12x^2 \]
\[ 0 = 12(8x - 8x + 12) \]
\[ 0 = 12(x - 6)(x - 2) \]
\[ x = 6 \text{ or } x = 2 \]

\[ \uparrow \text{Extraneous} \]
4. An open box (no lid) with a square base has a volume of 4 cubic feet. What dimensions will minimize the surface area?

\[ V = x \cdot x \cdot y \]
\[ 4 = x^2 \cdot y \]
\[ y = \frac{4}{x^2} \]
\[ y = \frac{4}{(2)^2} = 1 \text{ ft.} \]

\[ S_{A_{\text{min}}} = x^2 + 4x \cdot y \]
\[ S_{A_{\text{min}}} = x^2 + 4x \cdot \left( \frac{4}{x^2} \right) \]
\[ S_{A_{\text{min}}} = x^2 + \frac{16}{x} \]
\[ S_{A'} = 2x - \frac{16}{x^2} \]
\[ 0 = 2x - \frac{16}{x^2} \]
\[ \frac{16}{x^2} = 2x \]
\[ 2x^3 = 16 \]
\[ x^3 = 8 \]
\[ x = 2 \text{ ft.} \]

2 ft. \times 2 \text{ ft.} \times 1 \text{ ft.}

5. A sphere has a radius of 6 feet. What are the dimensions of the cylinder with the greatest volume that can be inscribed in the sphere? What is the maximum volume of this cylinder?

\[ r = \sqrt{3(6-x)^2} \]
\[ h = 2x \]

\[ r = \sqrt{3(6-12)} \]
\[ r = \sqrt{24} \text{ ft.} \]
\[ h = 2(2\sqrt{3}) \]
\[ h = 4\sqrt{3} \text{ ft.} \]

\[ V_{\text{max}} = \pi r^2 h \]
\[ V_{\text{max}} = \pi \left( \sqrt{3(6-x)^2} \right)^2 (2x) \]
\[ V_{\text{max}} = 2\pi x (3x-6) \]
\[ V_{\text{max}} = 72\pi x - 12\pi x^3 \]
\[ V' = 72\pi - 36\pi x^2 \]
\[ 0 = 72\pi - 36\pi x^2 \]
\[ x^2 = 2 \]
\[ x = \pm 2\sqrt{3} \]

\[ V_{\text{max}} = \pi (2\sqrt{3})^2 (4\sqrt{3}) \]
\[ V_{\text{max}} = (24\pi)(4\sqrt{3}) \]
\[ V_{\text{max}} = 96\pi\sqrt{3} \text{ ft.}^3 \]